

EFFICIENT APPROXIMATION OF THE SOLUTION OF CERTAIN NONLINEAR REACTION–DIFFUSION EQUATION II: THE CASE OF LARGE ABSORPTION

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ABSTRACT. We study the positive stationary solutions of a standard finite-difference discretization of the semilinear heat equation with nonlinear Neumann boundary conditions. We prove that, if the absorption is large enough, compared with the flux in the boundary, there exists a unique solution of such a discretization, which approximates the unique positive stationary solution of the “continuous” equation. Furthermore, we exhibit an algorithm computing an ε -approximation of such a solution by means of a homotopy continuation method. The cost of our algorithm is *linear* in the number of nodes involved in the discretization and the logarithm of the number of digits of approximation required.

1. INTRODUCTION

This article deals with the following semilinear heat equation with Neumann boundary conditions:

$$(1) \quad \begin{cases} u_t &= u_{xx} - g_1(u) & \text{in } (0, 1) \times [0, T), \\ u_x(1, t) &= \alpha g_2(u(1, t)) & \text{in } [0, T), \\ u_x(0, t) &= 0 & \text{in } [0, T), \\ u(x, 0) &= u_0(x) \geq 0 & \text{in } [0, 1], \end{cases}$$

where $g_1, g_2 \in \mathcal{C}^3(\mathbb{R})$ are analytic functions in $x = 0$ and α is a positive constant. The nonlinear heat equation models many physical, biological and engineering phenomena, such as heat conduction (see, e.g., [Can84, §20.3], [Pao92, §1.1]), chemical reactions and combustion (see, e.g., [BE89, §5.5], [Gri96, §1.7]), growth and migration of populations (see, e.g., [Mur02, Chapter 13], [Pao92, §1.1]), etc. In particular, “power-law” nonlinearities have long been of interest as a tractable prototype of general polynomial nonlinearities (see, e.g., [BE89, §5.5], [GK04, Chapter 7], [Lev90], [SGKM95], [Pao92, §1.1]).

The long-time behavior of the solutions of (1) has been intensively studied (see, e.g., [CFQ91], [GMW93], [Qui93], [Ros98], [FR01], [RT01], [AMTR02], [CQ04] and the references therein). In order to describe the dynamic behavior of the solutions of (1) it is usually necessary to analyze the behavior of the corresponding *stationary*

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solutions (see, e.g., [FR01], [CFQ91]), i.e., the positive solutions of the following two-point boundary-value problem:

$$(2) \quad \begin{cases} u_{xx} &= g_1(u) & \text{in } (0, 1), \\ u_x(1) &= \alpha g_2(u(1)), \\ u_x(0) &= 0. \end{cases}$$

The usual numerical approach to the solution of (1) consists of considering a second-order finite-difference discretization in the variable x , with a uniform mesh, keeping the variable t continuous (see, e.g., [BB98]). This semi-discretization in space leads to the following initial-value problem:

$$(3) \quad \begin{cases} u'_1 &= \frac{2}{h^2}(u_2 - u_1) - g_1(u_1), \\ u'_k &= \frac{1}{h^2}(u_{k+1} - 2u_k + u_{k-1}) - g_1(u_k), \quad (2 \leq k \leq n-1) \\ u'_n &= \frac{2}{h^2}(u_{n-1} - u_n) - g_1(u_n) + \frac{2\alpha}{h}g_2(u_n), \\ u_k(0) &= u_0(x_k), \quad (1 \leq k \leq n) \end{cases}$$

where $h := 1/(n-1)$ and x_1, \dots, x_n define a uniform partition of the interval $[0, 1]$. A similar analysis to that in [DM09] shows the convergence of the positive solutions of (3) to those of (1) and proves that every bounded solution of (3) tends to a stationary solution of (3), namely to a solution of

$$(4) \quad \begin{cases} 0 &= \frac{2}{h^2}(u_2 - u_1) - g_1(u_1), \\ 0 &= \frac{1}{h^2}(u_{k+1} - 2u_k + u_{k-1}) - g_1(u_k), \quad (2 \leq k \leq n-1) \\ 0 &= \frac{2}{h^2}(u_{n-1} - u_n) - g_1(u_n) + \frac{2\alpha}{h}g_2(u_n). \end{cases}$$

Hence, the dynamic behavior of the positive solutions of (3) is rather determined by the set of solutions $(u_1, \dots, u_n) \in (\mathbb{R}_{>0})^n$ of (4).

Very little is known concerning the study of the stationary solutions of (3) and the comparison between the stationary solutions of (3) and (1). In [FR01], [DM09] and [Dra10] there is a complete study of the positive solutions of (4) for the particular case $g_1(x) := x^p$ and $g_2(x) := x^q$, i.e., a complete study of the positive solutions of

$$(5) \quad \begin{cases} 0 &= \frac{2}{h^2}(u_2 - u_1) - u_1^p, \\ 0 &= \frac{1}{h^2}(u_{k+1} - 2u_k + u_{k-1}) - u_k^p, \quad (2 \leq k \leq n-1) \\ 0 &= \frac{2}{h^2}(u_{n-1} - u_n) - u_n^p + \frac{2\alpha}{h}u_n^q. \end{cases}$$

In [FR01] it is shown that there are spurious solutions of (4) for $q < p < 2q-1$, that is, positive solutions of (4) not converging to any solution of (2) as the mesh size h tends to zero.

In [DM09] and [Dra10] there is a complete study of (4) for $p > 2q-1$ and $p < q$. In these articles it is shown that in such cases there exists exactly one positive real solution. Furthermore, a numeric algorithm solving a given instance of the problem under consideration with $n^{O(1)}$ operations is proposed. In particular, the algorithm of [Dra10] has linear cost in n , that is, this algorithm gives a numerical approximation of the desired solution with $O(n)$ operations.

We observe that the family of systems (5) has typically an exponential number $O(p^n)$ of *complex* solutions ([DDM05]), and hence it is ill conditioned from the point of view of its solution by the so-called robust universal algorithms (cf. [Par00],

[CGH⁺03], [DMW09]). An example of such algorithms is that of general continuation methods (see, e.g., [AG90]). This shows the need of algorithms specifically designed to compute positive solutions of “structured” systems like (4).

Continuation methods aimed at approximating the real solutions of nonlinear systems arising from a discretization of two-point boundary-value problems for second-order ordinary differential equations have been considered in the literature (see, e.g., [ABSW06], [Duv90], [Wat80]). These works are usually concerned with Dirichlet problems involving an equation of the form $u_{xx} = f(x, u, u_x)$ for which the existence and uniqueness of solutions is known. Further, they focus on the existence of a suitable homotopy path rather on the cost of the underlying continuation algorithm. As a consequence, they do not seem to be suitable for the solution of (4). On the other hand, it is worth mentioning the analysis of [Kac02] on the complexity of shooting methods for two-point boundary value problems.

Let $g_1, g_2 \in \mathcal{C}^3(\mathbb{R})$ be analytic functions in $x = 0$ such that $g_i(0) = 0$, $g'_i(x) > 0$, $g''_i(x) > 0$ and $g'''_i(x) \geq 0$ for all $x > 0$ with $i = 1, 2$. We observe that g_1 and g_2 are a wide generalization of the monomial functions of system (5). Moreover, we shall assume throughout the paper that the functions $g := g_1/g_2$ and $G := G_1/g_2^2$ are strictly increasing, where G_1 is the primitive function of g_1 such that $G_1(0) = 0$, generalizing thus the relation $2q - 1 < p$ in (5). In this article we study the existence and uniqueness of the positive solutions of (4), and we obtain numerical approximations of these solutions using homotopy methods. In [?] there is a complete study of (4) for $g := g_1/g_2$ strictly decreasing. Furthermore, a similar analysis to that in [FR01] shows that there are spurious solutions of (4) for g strictly increasing and G strictly decreasing; i.e., the generalization of the relations $q < p < 2q - 1$ in (5). According to these remarks, we have a complete outlook about the existence and uniqueness of the positive solutions of (4).

1.1. Our contributions. In the first part of the article we prove that (4) has a unique positive solution, and we obtain upper and lower bounds for this solution independent of h , generalizing the results of [DM09].

In the second part of the article we exhibit an algorithm which computes an ε -approximation of the positive solution of (4). Such an algorithm is a continuation method that tracks the positive real path determined by the smooth homotopy obtained by considering (4) as a family of systems parametrized by α . Its cost is roughly of $n \log \log \varepsilon$ arithmetic operations, improving thus the exponential cost of general continuation methods.

The cost estimate of our algorithm is based on an analysis of the condition number of the corresponding homotopy path, which might be of independent interest. We prove that such a condition number can be bounded by a quantity independent of $h := 1/n$. This in particular implies that each member of the family of systems under consideration is significantly better conditioned than both an “average” dense system (see, e.g., [BCSS98, Chapter 13, Theorem 1]) and an “average” sparse system ([MR04, Theorem 1]).

1.2. Outline of the paper. In Section 2.2 we obtain upper and lower bounds for the coordinates of the positive solution of (4).

Section 2.3 is devoted to determine the number of positive solutions of (4). For this purpose, we prove that the homotopy of systems mentioned above is smooth

(Theorem 14). From this result we deduce the existence and uniqueness of the positive solutions of (4).

In Section 3 we obtain estimates on the condition number of the homotopy path considered in the previous section (Theorem 20). Such estimates are applied in Section 4 in order to estimate the cost of the homotopy continuation method for computing the positive solution of (4).

2. EXISTENCE AND UNIQUENESS OF STATIONARY SOLUTIONS

Let U_1, \dots, U_n be indeterminates over \mathbb{R} . Let g_1 and g_2 be two functions of class $\mathcal{C}^3(\mathbb{R})$ such that $g_i(0) = 0$, $g'_i(x) > 0$, $g''_i(x) > 0$ y $g'''_i(x) \geq 0$ for all $x > 0$ with $i = 1, 2$. As stated in the introduction, we are interested in the positive solutions of (4) for a given positive value of α , that is, in the positive solutions of the nonlinear system

$$(6) \quad \begin{cases} 0 &= -(U_2 - U_1) + \frac{h^2}{2}g_1(U_1), \\ 0 &= -(U_{k+1} - 2U_k + U_{k-1}) + h^2g_1(U_k), \quad (2 \leq k \leq n-1) \\ 0 &= -(U_{n-1} - U_n)\frac{h^2}{2}g_1(U_n) - h\alpha g_2(U_n), \end{cases}$$

for a given value $\alpha = \alpha^* > 0$, where $h := 1/(n-1)$. Observe that, as α runs through all possible values in $\mathbb{R}_{>0}$, one may consider (6) as a family of nonlinear systems parametrized by α , namely,

$$(7) \quad \begin{cases} 0 &= -(U_2 - U_1) + \frac{h^2}{2}g_1(U_1), \\ 0 &= -(U_{k+1} - 2U_k + U_{k-1}) + h^2g_1(U_k), \quad (2 \leq k \leq n-1) \\ 0 &= -(U_{n-1} - U_n)\frac{h^2}{2}g_1(U_n) - hAg_2(U_n), \end{cases}$$

where A is a new indeterminate.

2.1. Preliminary analysis. Let A, U_1, \dots, U_n be indeterminates over \mathbb{R} , set $U := (U_1, \dots, U_n)$ and denote by $F : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$ the nonlinear map defined by the right-hand side of (7). From the first $n-1$ equations of (7) we easily see that, for a given positive value $U_1 = u_1$, the (positive) values of U_2, \dots, U_n, A are uniquely determined. Therefore, letting U_1 vary, we may consider U_2, \dots, U_n, A as functions of U_1 , which are indeed recursively defined as follows:

$$(8) \quad \begin{aligned} U_1(u_1) &:= u_1, \\ U_2(u_1) &:= u_1 + \frac{h^2}{2}g_1(u_1), \\ U_{k+1}(u_1) &:= 2U_k(u_1) - U_{k-1}(u_1) + h^2g_1(U_k(u_1)), \quad (2 \leq k \leq n-1), \\ A(u_1) &:= \left(\frac{1}{h}(U_n - U_{n-1})(u_1) + \frac{h^2}{2}g_1(U_n(u_1)) \right) / g_2(U_n(u_1)). \end{aligned}$$

Arguing recursively, one deduces the following lemma (cf. [DDM05, Remark 20]).

Lemma 1. *For any $u_1 > 0$, the following assertions hold:*

- (i) $(U_k - U_{k-1})(u_1) = h^2 \left(\frac{1}{2}g_1(u_1) + \sum_{j=2}^{k-1} g_1(U_j(u_1)) \right) > 0$,
- (ii) $U_k(u_1) = u_1 + h^2 \left(\frac{k-1}{2}g_1(u_1) + \sum_{j=2}^{k-1} (k-j)g_1(U_j(u_1)) \right) > 0$,
- (iii) $(U'_k - U'_{k-1})(u_1) = h^2 \left(\frac{1}{2}g'_1(u_1) + \sum_{j=2}^{k-1} g'_1(U_j(u_1))U'_j(u_1) \right) > 0$,
- (iv) $U'_k(u_1) = 1 + h^2 \left(\frac{k-1}{2}g'_1(u_1) + \sum_{j=2}^{k-1} (k-j)g'_1(U_j(u_1))U'_j(u_1) \right) > 1$,

for $2 \leq k \leq n$.

As in [?] we have the following lemma. This result is important for the existence and uniqueness of the positive solutions of (7).

Lemma 2. *For any $u_1 > 0$, the following assertions hold:*

- (i) $\left(\frac{U_k - U_{k-1}}{g_1(U_k)}\right)'(u_1) < 0$,
- (ii) $\left(\frac{U_k - U_1}{g_1(U_k)}\right)'(u_1) < 0$,
- (iii) $\left(\frac{U_k - U_{k-1}}{U_k - U_1}\right)'(u_1) \geq 0$,
- (iv) $\left(\frac{g_1(U_k)}{g_1(U_1)}\right)'(u_1) > 0$,

for $2 \leq k \leq n$.

The next result studies the monotony of certain relations between g_1 and g_2 .

Lemma 3. *Let G_1 be the primitive function of g_1 such that $G_1(0) = 0$. If $x > 0$, then:*

- (i) $\left(\frac{g_1^2}{G_1}\right)'(x) > 0$.
- (ii) If $\left(\frac{G_1}{g_2^2}\right)'(x) > 0$, then $\left(\frac{g_1}{g_2}\right)'(x) > 0$.
- (iii) If exists $d \in [0, 1)$ such that $(\ln(G_1^d(x)/g_2^2(x)))' \geq 0$, then $\left(\frac{G_1}{g_2^2}\right)'(x) > 0$.

Proof. Since g_1 is a positive and strictly convex function in $\mathbb{R}_{>0}$ and $g_1(0) = 0$, we have that

$$(9) \quad \frac{g_1^2(x)}{2} = \int_0^x g_1(t)g_1'(t)dt < g_1'(x) \int_0^x g_1(t)dt = g_1'(x)G_1(x).$$

Multiplying both sides by $g_1(x)$, we obtain

$$\frac{g_1(x)}{G_1(x)} < \frac{2g_1(x)g_1'(x)}{g_1^2(x)},$$

which proves (i).

Now suppose that $(G_1/g_2^2)'(x) > 0$, then

$$\frac{2g_2(x)g_2'(x)}{g_2^2(x)} < \frac{g_1(x)}{G_1(x)}.$$

Combining this inequality with (9), we obtain

$$\frac{g_2'(x)}{g_2(x)} < \frac{g_1^2(x)}{2G_1(x)g_1'(x)} \frac{g_1'(x)}{g_1(x)} \leq \frac{g_1'(x)}{g_1(x)},$$

and (ii) is proved.

Finally, suppose that exists $d \in [0, 1)$ such that $(\ln(G_1^d(x)/g_2^2(x)))' \geq 0$, we deduce

$$\begin{aligned} 0 \leq (\ln(G_1^d(x)/g_2^2(x)))' &= d(\ln(G_1(x)))' - (\ln(g_2^2(x)))' \\ &= (\ln(G_1(x)))' \left(d - \frac{(\ln(g_2^2(x)))'}{(\ln(G_1(x)))'} \right). \end{aligned}$$

Since $(\ln(G_1(x)))' = g_1(x)/G_1(x) > 0$, we have that

$$\frac{(\ln(g_2^2(x)))'}{(\ln(G_1(x)))'} \leq d < 1.$$

From this inequality, we obtain

$$(10) \quad \frac{2g_2'(x)g_2(x)}{g_2^2(x)} = (\ln(g_2^2(x)))' < (\ln(G_1(x)))' = \frac{g_1(x)}{G_1(x)},$$

which completes the proof. \square

2.1.1. Analogy between discrete and continuous solutions. Set $u_k := U_k(u_1)$ for $2 \leq k \leq n$. The first step in our analysis of the positive solutions of (7) is to estimate the discrete derivative of the solution u of (2). Multiplying the identity $u'' = g_1(u)$ by u' and integrating over the interval $[0, x]$ it follows that

$$(11) \quad \frac{1}{2}u'(x)^2 = \int_0^x u'(s)u''(s)ds = \int_0^x u'(s)g_1(u(s))ds = G_1(u(x)) - G_1(u(0))$$

holds for any $x \in (0, 1)$, where G_1 is the primitive function of g_1 such that $G_1(0) = 0$. The following result shows that $\frac{1}{2}(\frac{u_m - u_{m-1}}{h})^2$, a discretization of $\frac{1}{2}u'(x)^2$, equals the trapezoidal rule applied to $\int_0^x g_1(u(s))u'(s)ds$ up to a certain error term.

Lemma 4. *For every $u_1 > 0$ and every $2 \leq m \leq n$, we have:*

$$(12) \quad \frac{1}{2}\left(\frac{u_m - u_{m-1}}{h}\right)^2 = \sum_{k=1}^{m-1} \frac{g_1(u_{k+1}) + g_1(u_k)}{2}(u_{k+1} - u_k) - \frac{g_1(u_1)}{4}(u_2 - u_1) - \frac{g_1(u_m)}{2}(u_m - u_{m-1}).$$

Proof. Fix $u_1 > 0$ and $2 \leq m \leq n$. For $m = 2$ the statement holds by (8). Next suppose that $m > 2$ holds. From (8), we deduce the following identities

$$\begin{aligned} \frac{2(u_2 - u_1)}{h^2} &= g_1(u_1), \\ \frac{u_{k+1} - 2u_k + u_{k-1}}{h^2} &= g_1(u_k), \quad (2 \leq k \leq m-1). \end{aligned}$$

Multiplying the first identity by $(u_2 - u_1)/4h$ and the k th identity by $(u_{k+1} - u_{k-1})/2h$, we obtain

$$\begin{aligned} \frac{1}{2h}\left(\frac{u_2 - u_1}{h}\right)^2 &= \frac{1}{4}\left(\frac{u_2 - u_1}{h}\right)g_1(u_1), \\ \frac{1}{2h}\left(\left(\frac{u_{k+1} - u_k}{h}\right)^2 - \left(\frac{u_k - u_{k-1}}{h}\right)^2\right) &= \frac{1}{2}\left(\frac{u_{k+1} - u_k}{h} + \frac{u_k - u_{k-1}}{h}\right)g_1(u_k), \end{aligned}$$

for $2 \leq k \leq m-1$. Note that $(u_{k+1} - u_{k-1})/2h$ is a numerical approximation of $u'((k-1)h)$ for $2 \leq k \leq m-1$. Adding these identities multiplied by h we obtain:

$$\begin{aligned} \frac{1}{2}\left(\frac{u_m - u_{m-1}}{h}\right)^2 &= \frac{g_1(u_1)}{4}(u_2 - u_1) + \sum_{k=2}^{m-1} \frac{g_1(u_k)}{2}(u_{k+1} - u_k + u_k - u_{k-1}) \\ &= \sum_{k=1}^{m-1} \frac{g_1(u_k) + g_1(u_{k+1})}{2}(u_{k+1} - u_k) - \frac{g_1(u_1)}{4}(u_2 - u_1) \\ &\quad - \frac{g_1(u_m)}{2}(u_m - u_{m-1}). \end{aligned}$$

This finishes the proof of the lemma. \square

Substituting x for 1 in (11) we obtain the following identity (cf. [CFQ91, §3]):

$$(13) \quad \frac{1}{2}\alpha^2 g_2^2(u(1)) = G_1(u(1)) - G_1(u(0)).$$

From this identity one easily deduces that $u(1)$ is determined in terms of $\nu_0 := u(0)$, say, $u(1) = f(\nu_0)$. Therefore, by (13) it is possible to restate (2) as an initial-value problem, namely

$$\begin{cases} u_{xx} &= g_1(u) & \text{in } (0, 1), \\ u(0) &= \nu_0, \\ u_x(0) &= 0, \end{cases}$$

where $\nu_0 > 0$ is the solution of the equation $u_x(1) = \alpha g_2(f(\nu_0))$. Our purpose is to obtain a discrete analogue of this identity, which will be crucial to determine the values u_1 of the positive solutions of (7).

Let $(\alpha, u) := (\alpha, u_1, \dots, u_n) \in (\mathbb{R}_{>0})^{n+1}$ be a solution of (7). From the last equation of (7) we obtain

$$\begin{aligned} \frac{1}{2}\left(\frac{u_n - u_{n-1}}{h}\right)^2 &= \frac{1}{2}\left(\alpha g_2(u_n) - \frac{h}{2}g_1(u_n)\right)^2 \\ &= \frac{1}{2}\alpha^2 g_2^2(u_n) + \frac{h}{2}g_1(u_n)\left(\frac{h}{2}g_1(u_n) - \alpha g_2(u_n)\right) - \frac{h^2}{8}g_1^2(u_n) \\ &= \frac{1}{2}\alpha^2 g_2^2(u_n) - \frac{h}{2}g_1(u_n)\left(\frac{u_n - u_{n-1}}{h}\right) - \frac{h^2}{8}g_1^2(u_n). \end{aligned}$$

Combining this identity with Lemma 4 we obtain

$$\frac{1}{2}\alpha^2 g_2^2(u_n) = \sum_{k=1}^{n-1} \frac{g_1(u_k) + g_1(u_{k+1})}{2}(u_{k+1} - u_k) - \frac{g_1(u_1)}{4}(u_2 - u_1) + \frac{h^2}{8}g_1^2(u_n).$$

Using the identity $g_1(u_1)(u_2 - u_1) = \frac{h^2}{2}g_1^2(u_1)$, we deduce that

$$\frac{1}{2}\alpha^2 g_2^2(u_n) - (G_1(u_n) - G_1(u_1)) = E + \frac{h^2}{8}(g_1^2(u_n) - g_1^2(u_1)),$$

holds, where G_1 is the primitive function of g_1 such that $G_1(0) = 0$ and E is defined as follow:

$$(14) \quad E := \sum_{k=1}^{n-1} \frac{g_1(u_k) + g_1(u_{k+1})}{2}(u_{k+1} - u_k) - (G_1(u_n) - G_1(u_1)).$$

It is easy to check that E is the error of the approximation by the trapezoidal rule of the integral of the function g_1 in the interval $[u_1, u_n]$, considering the subdivision of $[u_1, u_n]$ defined by the nodes u_1, \dots, u_n . Moreover, taking into account that g_1 is a convex function in $\mathbb{R}_{\geq 0}$, we easily conclude that $E \geq 0$ holds. Therefore, from the previous considerations we deduce the following proposition, which is the discrete version of (13).

Proposition 5. *Let $(\alpha, u) \in (\mathbb{R}_{>0})^{n+1}$ be a solution of (7). Then*

$$(15) \quad \frac{1}{2}\alpha^2 g_2^2(u_n) - (G_1(u_n) - G_1(u_1)) = E + \frac{h^2}{8}(g_1^2(u_n) - g_1^2(u_1)),$$

where G_1 is the primitive function of g_1 such that $G_1(0) = 0$, and E is defined as in (14). Furthermore, if we consider E as a function of u_1 according to (14), where $u_k := U_k(u_1)$ is defined as in (8) for $2 \leq k \leq n$, then E is a positive increasing function over $\mathbb{R}_{\geq 0}$.

Proof. For the above considerations, we only have to prove that E is an increasing function over $\mathbb{R}_{\geq 0}$.

We consider \bar{E} as a function of u_1 , where $u_k := U_k(u_1)$ is defined as in (8) for $2 \leq k \leq n$. If rewrite E as follows

$$(16) \quad E = \sum_{k=1}^{n-1} \left(\frac{g_1(u_k) + g_1(u_{k+1})}{2} (u_{k+1} - u_k) - (G_1(u_{k+1}) - G_1(u_k)) \right),$$

then it suffices to show that each term of the previous sum is an increasing function over $\mathbb{R}_{\geq 0}$. In fact, fix $1 \leq k \leq n-1$; the derivative of the k th term of (16) as function of u_1 is

$$\begin{aligned} & \frac{\partial}{\partial u_1} \left(\frac{g_1(u_k) + g_1(u_{k+1})}{2} (u_{k+1} - u_k) - (G_1(u_{k+1}) - G_1(u_k)) \right) = \\ &= \frac{g'_1(u_k)v'_k + g'_1(u_{k+1})v'_{k+1}}{2} (u_{k+1} - u_k) + \frac{g_1(u_k) + g_1(u_{k+1})}{2} (v'_{k+1} - v'_k) \\ & \quad - (g_1(u_{k+1})v'_{k+1} - g_1(u_k)v'_k), \end{aligned}$$

where $v'_k := U'_k(u_1)$ and $v'_{k+1} := U'_{k+1}(u_1)$. Adding and subtracting v'_k in each occurrence of v'_{k+1} we obtain

$$\begin{aligned} & \frac{\partial}{\partial u_1} \left(\frac{g_1(u_k) + g_1(u_{k+1})}{2} (u_{k+1} - u_k) - (G_1(u_{k+1}) - G_1(u_k)) \right) = \\ &= \left(\frac{g'_1(u_k) + g'_1(u_{k+1})}{2} (u_{k+1} - u_k) - (g_1(u_{k+1}) - g_1(u_k)) \right) v'_k \\ & \quad + \left(\frac{g'_1(u_{k+1})}{2} (u_{k+1} - u_k) - \frac{g_1(u_{k+1}) - g_1(u_k)}{2} \right) (v'_{k+1} - v'_k) \\ &= \left(\frac{g'_1(u_k) + g'_1(u_{k+1})}{2} (u_{k+1} - u_k) - (g_1(u_{k+1}) - g_1(u_k)) \right) v'_k \\ & \quad + \left(\frac{g'_1(u_{k+1})}{2} - \frac{g'_1(\xi_k)}{2} \right) (u_{k+1} - u_k) (v'_{k+1} - v'_k), \end{aligned}$$

where $\xi_k \in (u_k, u_{k+1})$ is obtained after applying the Mean Value Theorem to $g_1(u_{k+1}) - g_1(u_k)$. It is easy to check that

$$\frac{g'_1(u_k) + g'_1(u_{k+1})}{2} (u_{k+1} - u_k) - (g_1(u_{k+1}) - g_1(u_k))$$

is the error of the approximation by the trapezoidal rule of the integral of the function g'_1 in the interval $[u_k, u_{k+1}]$, and the convexity of g'_1 ensures their positivity. In the other hand, since g'_1 is increasing, we have that $g'_1(u_{k+1}) - g'_1(\xi_k) \geq 0$. Finally, from Lema 1, $(v'_{k+1} - v'_k)$, $(u_{k+1} - u_k)$ and v'_k are positive numbers for $u_1 \in \mathbb{R}_{\geq 0}$. Therefore, the k th term of (16) is increasing over $\mathbb{R}_{\geq 0}$ for $1 \leq k \leq n-1$, which completes the proof. \square

2.2. Bounds for the positive solutions. In this section we show bounds for the positive solutions of (7). More precisely, we find an interval containing the positive solutions of (7) whose endpoints only depend on α . These bounds will allow us to establish an efficient procedure of approximation of this solution.

Let $g : \mathbb{R}_{>0} \rightarrow \mathbb{R}_{>0}$ and $G : \mathbb{R}_{>0} \rightarrow \mathbb{R}_{>0}$ be the functions defined by

$$(17) \quad g(x) := \frac{g_1}{g_2}(x),$$

and

$$(18) \quad G(x) := \frac{G_1}{g_2^2}(x),$$

where G_1 is the primitive function of g_1 such that $G_1(0) = 0$.

As in [?, Lemma 7] we have the following result

Lemma 6. *Let $(\alpha, u) \in (\mathbb{R}_{>0})^{n+1}$ be a solution of (7) for $A = \alpha$. Then*

$$\alpha g_2(u_n) < g_1(u_n).$$

From Lemma 6 we obtain the following corollary.

Corollary 7. *Let $(\alpha, u) \in (\mathbb{R}_{>0})^{n+1}$ be a solution of (7) for $A = \alpha$. If the function g defined in (17) is surjective and strictly increasing, then*

$$u_n > g^{-1}(\alpha).$$

Let $(\alpha, u) \in (\mathbb{R}_{>0})^{n+1}$ be a solution of (7) for $A = \alpha$. As in [?, Lemma 9] we obtain an upper bound of u_n in terms of u_1 and α .

Lemma 8. *Let $(\alpha, u) \in (\mathbb{R}_{>0})^{n+1}$ be a solution of (7) for $A = \alpha$, and let $C(\alpha)$ be an upper bound of u_n . Then $u_n < e^M u_1$ holds, with $M := g'_1(C(\alpha))$.*

The next lemma shows a lower bound of u_1 in terms of α .

Lemma 9. *Let $(\alpha, u) \in (\mathbb{R}_{>0})^{n+1}$ be a solution of (7) for $A = \alpha$, and let $C(\alpha)$ be an upper bound of u_n . If the function g defined in (17) is surjective and strictly increasing, then*

$$u_1 > \frac{g^{-1}(\alpha)}{e^M}$$

holds, where $M := g'_1(C(\alpha))$.

Proof. From Lemma 8 and Corollary 7 we deduce

$$g^{-1}(\alpha) < u_n < e^M u_1,$$

which immediately implies the statement of the lemma. \square

In the next lemma we obtain another upper bound of u_n in terms of u_1 and α . This upper bound will allow us to find an upper bound of u_n in terms of α .

Lemma 10. *Let $(\alpha, u) \in (\mathbb{R}_{>0})^{n+1}$ be a solution of (7) for $A = \alpha$. Then*

$$G(u_n) < G(u_1) + \frac{\alpha^2}{2},$$

where G is defined in (18).

Moreover, if G is surjective and strictly increasing, then

$$u_n < G^{-1}\left(G(u_1) + \frac{\alpha^2}{2}\right).$$

Proof. From Proposition 5 and Lemma 1, we deduce the following inequality

$$G_1(u_n) - \frac{\alpha^2}{2} g_2^2(u_n) < G_1(u_1),$$

where G_1 is the primitive function of g_1 such that $G_1(0) = 0$. Dividing for $g_2^2(u_n)$, we obtain

$$G(u_n) - \frac{\alpha^2}{2} < \frac{G_1(u_1)}{g_2^2(u_n)}.$$

Since g_2^2 is an increasing function, we conclude that

$$G(u_n) - \frac{\alpha^2}{2} < \frac{G_1(u_1)}{g_2^2(u_n)} \leq G(u_1),$$

which prove the first part of the lemma.

Now, suppose that G is surjective and strictly increasing, then G is an invertible function and their inverse is strictly increasing. Combining this remark with the last inequality, we obtain

$$u_n < G^{-1}\left(G(u_1) + \frac{\alpha^2}{2}\right),$$

and the proof is complete. \square

From this lemma we obtain upper bounds for u_1 and u_n in terms of α .

Proposition 11. *Let $(\alpha, u) \in (\mathbb{R}_{>0})^{n+1}$ be a solution of (7) and let g and G be the functions defined in (17) and (18) respectively. Suppose that*

- *exists $d \in [0, 1)$ such that $(\ln(G_1^d(x)/g_2^2(x)))' \geq 0$ for all $x > 0$,*
- *$G''(x) \geq 0$ for all $x > 0$,*

hold, where G_1 is the primitive function of g_1 such that $G_1(0) = 0$. Then

$$g^2(u_1) < \frac{\alpha^2}{1-d}.$$

Moreover, if g and G are surjective functions, then

$$u_1 < g^{-1}\left(\frac{\alpha}{\sqrt{1-d}}\right),$$

and

$$u_n < G^{-1}\left(G\left(g^{-1}\left(\frac{\alpha}{\sqrt{1-d}}\right)\right) + \frac{\alpha^2}{2}\right).$$

Proof. Combining Lemma 8 and the Mean Value Theorem, we deduce that exists $\xi > 0$ between u_1 and u_n such that

$$G'(\xi)(u_n - u_1) = G(u_n) - G(u_1) < \frac{\alpha^2}{2}.$$

By Lemma 1(ii), we have that

$$(u_n - u_1) = h^2\left(\frac{n-1}{2}g_1(u_1) + \sum_{j=2}^{n-1}(n-j)g_1(u_j)\right) > \frac{g_1(u_1)}{2} > 0.$$

Combining both inequalities, we obtain that

$$G'(\xi)\frac{g_1(u_1)}{2} < G'(\xi)(u_n - u_1) < \frac{\alpha^2}{2}.$$

Since $G''(x) \geq 0$ for all $x > 0$, we see that G' is an increasing function. Furthermore, we have the following inequality:

$$\begin{aligned}
 \alpha^2 > G'(u_1)g_1(u_1) &= \left(\frac{g_1(u_1)g_2^2(u_1) - G_1(u_1)2g_2(u_1)g_2'(u_1)}{g_2^4(u_1)} \right) g_1(u_1) \\
 (19) \qquad \qquad \qquad &= \left(1 - \frac{G_1(u_1)2g_2(u_1)g_2'(u_1)}{g_1(u_1)g_2^2(u_1)} \right) \frac{g_1^2(u_1)}{g_2^2(u_1)} \\
 &= \left(1 - \frac{(\ln(g_2^2(u_1)))'}{(\ln(G_1(u_1)))'} \right) \frac{g_1^2(u_1)}{g_2^2(u_1)}.
 \end{aligned}$$

Taking into account the first condition of the statement, we deduce that

$$\begin{aligned}
 0 \leq (\ln(G_1^d(x)/g_2^2(x)))' &= d(\ln(G_1(x)))' - (\ln(g_2^2(x)))' \\
 &= (\ln(G_1(x)))' \left(d - \frac{(\ln(g_2^2(x)))'}{(\ln(G_1(x)))'} \right).
 \end{aligned}$$

Since $(\ln(G_1(x)))' = g_1(x)/G_1(x) > 0$, we conclude that

$$(20) \qquad \qquad \qquad \frac{(\ln(g_2^2(x)))'}{(\ln(G_1(x)))'} \leq d.$$

Combining (19) and (20), we obtain

$$(21) \qquad \qquad \qquad \alpha^2 > (1-d) \frac{g_1^2(u_1)}{g_2^2(u_1)} = (1-d)g^2(u_1),$$

and the first assertion of the proposition is proved.

Now, suppose that g and G are surjective functions. From (3), we have that g and G are strictly increasing functions. Combining this remark with (21) and Lemma 10, we obtain the desired upper bounds for u_1 and u_n . \square

Combining Proposition 11 and Lemma 9 we obtain the following result.

Lemma 12. *Let $(\alpha, u) \in (\mathbb{R}_{>0})^{n+1}$ be a solution of (7) and let g and G be the functions defined in (17) and (18) respectively. Suppose that*

- G and g are surjective functions,
- exists $d \in [0, 1)$ such that $(\ln(G_1^d(x)/g_2^2(x)))' \geq 0$ for all $x > 0$,
- $G''(x) \geq 0$ for all $x > 0$,

hold, where G_1 is the primitive function of g_1 such that $G_1(0) = 0$. Then

$$u_1 < g^{-1}(\alpha \widehat{C}(\alpha)),$$

where

$$\widehat{C}(\alpha) := 1 + \frac{g_2'(C_1(\alpha))\alpha^2}{2g_2(g^{-1}(\alpha)/e^M)G'(g^{-1}(\alpha)/e^M)},$$

with

$$C_1(\alpha) := G^{-1}\left(G\left(g^{-1}\left(\frac{\alpha}{\sqrt{1-d}}\right)\right) + \frac{\alpha^2}{2}\right),$$

and $M := g_1'(C_1(\alpha))$. Furthermore,

$$u_n < G^{-1}\left(G\left(g^{-1}\left(\alpha \widehat{C}(\alpha)\right)\right) + \frac{\alpha^2}{2}\right).$$

Proof. Let $(\alpha, u) \in (\mathbb{R}_{>0})^{n+1}$ be a solution of (7). From Lemmas 1 and 10 we deduce the inequalities

$$\begin{aligned} \bullet \quad & g_1(u_1) < h\left(\frac{1}{2}g_1(u_1) + g_1(u_2) + \cdots + g_1(u_{n-1}) + \frac{1}{2}g_1(u_n)\right) = \alpha g_2(u_n), \\ \bullet \quad & u_n < G^{-1}\left(G(u_1) + \frac{\alpha^2}{2}\right). \end{aligned}$$

Combining both inequalities we obtain

$$g(u_1) < \alpha \frac{g_2(u_n)}{g_2(u_1)} < \alpha \frac{g_2\left(G^{-1}\left(G(u_1) + \frac{\alpha^2}{2}\right)\right)}{g_2(u_1)}.$$

Since g'_2 is an increasing function in $\mathbb{R}_{>0}$ and $(G^{-1})'$ is a decreasing function in $\mathbb{R}_{>0}$, by the Mean Value Theorem, we obtain the following estimates

$$\begin{aligned} g(u_1) &< \alpha \frac{g_2(u_1) + g'_2\left(G^{-1}\left(G(u_1) + \frac{\alpha^2}{2}\right)\right)\left(G^{-1}\left(G(u_1) + \frac{\alpha^2}{2}\right) - u_1\right)}{g_2(u_1)} \\ &< \alpha \frac{g_2(u_1) + g'_2\left(G^{-1}\left(G(u_1) + \frac{\alpha^2}{2}\right)\right)(G^{-1})'(G(u_1))\frac{\alpha^2}{2}}{g_2(u_1)} \\ &< \alpha \left(1 + \frac{g'_2\left(G^{-1}\left(G(u_1) + \frac{\alpha^2}{2}\right)\right)\alpha^2}{2g_2(u_1)G'(u_1)}\right). \end{aligned}$$

From Proposition 11 and Lemma 9 we conclude that

$$u_1 < g^{-1}(\alpha \hat{C}(\alpha)),$$

where

$$\hat{C}(\alpha) := 1 + \frac{g'_2(C_1(\alpha))\alpha^2}{2g_2(g^{-1}(\alpha)/e^M)G'(g^{-1}(\alpha)/e^M)},$$

with $M := g'_1(C_1(\alpha))$ and

$$C_1(\alpha) := G^{-1}\left(G\left(g^{-1}\left(\frac{\alpha}{\sqrt{1-d}}\right)\right) + \frac{\alpha^2}{2}\right).$$

Combining this remark with Lemma 10 we obtain

$$u_n < G^{-1}\left(G\left(g^{-1}\left(\alpha \hat{C}(\alpha)\right)\right) + \frac{\alpha^2}{2}\right),$$

which immediately implies the statement of the lemma. \square

2.3. Existence and uniqueness. Let $P : (\mathbb{R}_{>0})^2 \rightarrow \mathbb{R}$ be the nonlinear map defined by

$$(22) \quad P(\alpha, u_1) := \frac{1}{h}(U_{n-1}(u_1) - U_n(u_1)) - \frac{h}{2}g_1(U_n(u_1)) + \alpha g_2(U_n(u_1)).$$

Observe that $P(A, U_1) = 0$ represents the minimal equation satisfied by the coordinates (α, u_1) of any (complex) solution of the nonlinear system (7). Therefore, for fixed $\alpha \in \mathbb{R}_{>0}$, the positive roots of $P(\alpha, U_1)$ are the values of u_1 we want to obtain. Furthermore, from the parametrizations (8) of the coordinates u_2, \dots, u_n of a given solution $(\alpha, u_1, \dots, u_n) \in (\mathbb{R}_{>0})^{n+1}$ of (7) in terms of u_1 , we conclude that the number of positive roots of $P(\alpha, U_1)$ determines the number of positive solutions of (7) for such a value of α .

Since $P(A, U_1)$ is a continuous function in $(\mathbb{R}_{>0})^2$, as in [?, Proposition 4] we have the following result:

Proposition 13. Fix $\alpha > 0$ and $n \in \mathbb{N}$. If the function g defined in (17) is surjective, then (7) has a positive solution with $A = \alpha$.

In order to establish the uniqueness, we prove that the homotopy path that we obtain by moving the parameter α in $\mathbb{R}_{>0}$ is smooth. For this purpose, we show that the rational function $A(U_1)$ implicitly defined by the equation $P(A, U_1) = 0$ is increasing. We observe that an explicit expression for this function in terms of U_1 is obtained in (8).

Theorem 14. Let $\mathcal{A} > 0$ be a given constant and let $A(U_1)$ be the rational function of (8). Let g and G be the functions defined in (17) and (18) respectively. Suppose that

- G and g are surjective functions,
- exists $d \in [0, 1)$ such that $(\ln(G_1^d(x)/g_2^2(x)))' \geq 0$ for all $x > 0$,
- $G''(x) \geq 0$ for all $x > 0$,

hold, where G_1 is the primitive function of g_1 such that $G_1(0) = 0$. Then there exists $M(\mathcal{A}) > 0$ such that the condition $A'(u_1) > 0$ is satisfied for $n > 1 + M(\mathcal{A})/(2 - 2d)$ and $u_1 \in A^{-1}((0, \mathcal{A}]) \cap \mathbb{R}_{>0}$.

Proof. Let U_1, U_2, \dots, U_n, A be the functions defined in (8). For $u_1 > 0$, we denote by $I(u_1) := G_1(U_n(u_1)) - G_1(U_1(u_1))$ the integral of the function g_1 in $[U_1(u_1), U_n(u_1)]$, and by $T(u_1)$ the trapezoidal rule applied to $I(u_1)$, with the nodes $U_1(u_1), U_2(u_1), \dots, U_n(u_1)$. More precisely, T is define as follows:

$$T := \sum_{k=1}^{n-1} \frac{g_1(U_{k+1}) + g_1(U_k)}{2} (U_{k+1} - U_k).$$

Finally, set $E := T - I$. Combining Proposition 5 and the convexity of g_1 , we deduce that $E > 0$ and $E' > 0$ in $\mathbb{R}_{>0}$, where E' represent the derivative of E with respect of u_1 .

According to Proposition 5, U_1, U_2, \dots, U_n, A satisfy the discrete version (15) of the energy conservation law (11). Dividing both sides of (15) by $G_1(U_n)$ we obtain the following identities:

$$(23) \quad \frac{1}{2} A^2 \frac{g_2^2(U_n)}{G_1(U_n)} = \frac{T}{G_1(U_n)} + \frac{h^2}{8} \frac{g_1^2(U_n)}{G_1(U_n)} \left(1 - \frac{g_1^2(U_1)}{g_1^2(U_n)}\right).$$

Taking derivatives with respect to U_1 at both sides of (23), we have

$$(24) \quad \begin{aligned} & AA' \frac{g_2^2(U_n)}{G_1(U_n)} + \frac{A^2}{2} \left(\frac{g_2^2(U_n)}{G_1(U_n)} \right)' = \\ & = \left(\frac{T}{G_1(U_n)} \right)' + \frac{h^2}{8} \left(\frac{g_1^2(U_n)}{G_1(U_n)} \right)' \left(1 - \frac{g_1^2(U_1)}{g_1^2(U_n)}\right) - \frac{h^2}{8} \frac{g_1^2(U_n)}{G_1(U_n)} \left(\frac{g_1^2(U_1)}{g_1^2(U_n)} \right)'. \end{aligned}$$

Let $u_1 \in A^{-1}((0, \mathcal{A}]) \cap \mathbb{R}_{>0}$. By Lema 1, $U_i(u_1)$ and $U_i'(u_1)$ are positive for $1 \leq i \leq n$. Furthermore, g_1, g_2, g, G and G_1 are positive and increasing functions in $\mathbb{R}_{>0}$. Throughout the demonstration we will use these conditions repeatedly.

From Lemmas 3 and 2 we deduce that $g_1^2(U_n)/G_1(U_n)$ is an increasing function and $g_1^2(U_1)/g_1^2(U_n)$ is a decreasing function. Combining these remarks with (24) we obtain

$$\left(AA' \frac{g_2^2(U_n)}{G_1(U_n)} \right)(u_1) > \left(\left(\frac{T}{G_1(U_n)} \right)' - \frac{A^2}{2} \left(\frac{g_2^2(U_n)}{G_1(U_n)} \right)' \right)(u_1).$$

we see that the inequality above may be rewritten in the form

$$(25) \quad \left(AA' \frac{g_2^2(U_n)}{G_1(U_n)} \right)_{(u_1)} > \left(\frac{I^2}{G_1^2(U_n)} \left(\frac{T}{I} \right)' + \frac{G_1^2(U_1)}{G_1^2(U_n)} \left(\frac{T}{G_1(U_1)} \right)' \right)_{(u_1)} - \left(\frac{A^2}{2} \left(\frac{g_2^2(U_n)}{G_1(U_n)} \right)' \right)_{(u_1)}.$$

We claim that $(T/G_1(U_1))'(u_1) > 0$. Indeed, by Lemmas 1 and 2 we have that

$$\left(\frac{T}{g_1^2(U_1)} \right)'_{(u_1)} = \left(\sum_{k=1}^{n-1} \frac{g_1(U_{k+1}) + g_1(U_k)}{2g_1(U_1)} h^2 \left(\frac{1}{2} + \sum_{j=2}^k \frac{g_1(U_j)}{g_1(U_1)} \right) \right)'_{(u_1)} > 0.$$

Combining this result with Lemma 3, we conclude that

$$\left(\frac{T}{G_1(U_1)} \right)'_{(u_1)} = \left(\frac{g_1^2(U_1)}{G_1(U_1)} \frac{T}{g_1^2(U_1)} \right)'_{(u_1)} > 0.$$

Combining the claim above with (25) we deduce that

$$(26) \quad \left(AA' \frac{g_2^2(U_n)}{G_1(U_n)} \right)_{(u_1)} > \left(\frac{I^2}{G_1^2(U_n)} \left(\frac{T}{I} \right)' - \frac{A^2}{2} \left(\frac{g_2^2(U_n)}{G_1(U_n)} \right)' \right)_{(u_1)}.$$

In order to prove the positivity of $A'(u_1)$, we rewrite the right side of (26).

$$\begin{aligned} & \left(\frac{I^2}{G_1^2(U_n)} \left(\frac{T}{I} \right)' - \frac{A^2}{2} \left(\frac{g_2^2(U_n)}{G_1(U_n)} \right)' \right)_{(u_1)} = \\ & = \left(\frac{T'I - TI'}{G_1^2(U_n)} + \frac{A^2}{2} \frac{g_2^2(U_n)(G_1(U_n))' - (g_2^2(U_n))'G_1(U_n)}{G_1^2(U_n)} \right)_{(u_1)} \\ & = \left(\frac{E'I - EI'}{G_1^2(U_n)} + \frac{A^2 g_2^2(U_n)(G_1(U_n))'}{2G_1^2(U_n)} \left(1 - \frac{(g_2^2(U_n))'G_1(U_n)}{g_2^2(U_n)(G_1(U_n))'} \right) \right)_{(u_1)}. \end{aligned}$$

Since there exists $d \in [0, 1)$ such that $(\ln(G_1^d(x)/g_2^2(x)))' \geq 0$ for all $x > 0$, we have that

$$\left(1 - \frac{(g_2^2(U_n))'G_1(U_n)}{g_2^2(U_n)(G_1(U_n))'} \right)_{(u_1)} > 1 - d.$$

Furthermore, by the positivity of $E'(u_1)$ and the definition of $I(u_1)$, we conclude that $(E'I - EI')(u_1) > -E(u_1)(G(U_n))'(u_1)$. From these inequalities, we deduce that

$$\begin{aligned} & \left(\frac{I^2}{G_1^2(U_n)} \left(\frac{T}{I} \right)' - \frac{A^2}{2} \left(\frac{g_2^2(U_n)}{G_1(U_n)} \right)' \right)_{(u_1)} > \\ & > \left(\frac{(G_1(U_n))'}{G_1^2(U_n)} \left((1-d) \frac{A^2 g_2^2(U_n)}{2} - E \right) \right)_{(u_1)}. \end{aligned}$$

Combining these remarks with (26), we obtain

$$\left(AA' \frac{g_2^2(U_n)}{G_1(U_n)} \right)_{(u_1)} > \left(\frac{(G_1(U_n))'}{G_1^2(U_n)} \left((1-d) \frac{A^2 g_2^2(U_n)}{2} - E \right) \right)_{(u_1)}.$$

From (15), it follows that

$$(27) \quad \left(AA' \frac{g_2^2(U_n)}{G_1(U_n)} \right)_{(u_1)} > \left(\frac{(G_1(U_n))'}{G_1^2(U_n)} \frac{(1-d)A^2 g_2^2(U_n)}{4} \right)_{(u_1)} \\ + \left(\frac{(G_1(U_n))'}{G_1^2(U_n)} \left(\frac{1-d}{2} T + \frac{1-d}{2} \frac{h^2}{8} (g_1^2(U_n) - g_1^2(U_1)) - E \right) \right)_{(u_1)}.$$

If we prove that the second term of the right side of (27) is positive, we obtain that

$$(28) \quad \left(AA' \frac{g_2^2(U_n)}{G_1(U_n)} \right)_{(u_1)} > \left(\frac{(G_1(U_n))'}{G_1^2(U_n)} \frac{(1-d)A^2 g_2^2(U_n)}{4} \right)_{(u_1)} > 0,$$

which immediately implies the statement of the theorem. Therefore, it suffices to show that

$$\left(\frac{(G_1(U_n))'}{G_1^2(U_n)} \left(\frac{1-d}{2} T + \frac{1-d}{2} \frac{h^2}{8} (g_1^2(U_n) - g_1^2(U_1)) - E \right) \right)_{(u_1)} > 0.$$

Since g_1 is an increasing function, we only need to show that

$$\frac{1-d}{2} T(u_1) - E(u_1) = \sum_{k=1}^{n-1} \left(\frac{1-d}{2} T_k(u_1) - E_k(u_1) \right) > 0,$$

where

$$T_k := \frac{g_1(U_{k+1}) + g_1(U_k)}{2} (U_{k+1} - U_k), \\ E_k := T_k - I_k,$$

with $I_k := G_1(U_{k+1}) - G_1(U_k)$. Note that, for $u_1 > 0$, $I_k(u_1)$ is the integral of g_1 in $[U_k(u_1), U_{k+1}(u_1)]$, $T_k(u_1)$ is the trapezoidal rule applied to $I_k(u_1)$ and $E_k(u_1)$ is the error of such approximation.

In order to prove that $(1-d)T(u_1)/2 - E(u_1) > 0$, we show that

$$(29) \quad \frac{1-d}{2} T_k(u_1) - E_k(u_1) > 0$$

holds for $1 \leq k \leq n-1$. By [?], we have that

$$E_k(u_1) \leq \left(\frac{g_1'(U_{k+1}) + g_1'(U_k)}{8} (U_{k+1} - U_k)^2 \right)_{(u_1)}.$$

From Lemma 1 and the monotonicity of g_1' , we deduce that

$$E_k(u_1) \leq \left(\frac{g_1'(U_n)}{4} (U_{k+1} - U_k)^2 \right)_{(u_1)} \\ \leq \left(h^2 \frac{g_1'(U_n)}{4} \sum_{j=1}^k \frac{g_1(U_{j+1}) + g_1(U_j)}{2} (U_{k+1} - U_k) \right)_{(u_1)} \\ \leq \left(h \frac{g_1'(U_n)}{4} \frac{g_1(U_{k+1}) + g_1(U_k)}{2} (U_{k+1} - U_k) \right)_{(u_1)} \\ = \left(h \frac{g_1'(U_n)}{4} T_k \right)_{(u_1)}.$$

Thus, we see that (29) is satisfied if the inequality

$$(30) \quad h \frac{g'_1(U_n)(u_1)}{4} \leq \frac{1-d}{2}$$

holds. From Lemma 12 and the monotonicity of g'_1 , we deduce that there exists a constant $M(\mathcal{A}) > 0$ independent of h such that $g'_1(U_n)(u_1) \leq M(\mathcal{A})$ for $u_1 \in A^{-1}((0, \mathcal{A}]) \cap \mathbb{R}_{>0}$. This shows that a sufficient condition for the fulfillment of (30), and thus of $A'(U_1) > 0$, is that $n-1 \geq M(\mathcal{A})/(2-2d)$ holds. This finishes the proof of the theorem. \square

In order to prove the uniqueness of positive solutions of (7), we still need a result on the structure of the inverse image of A on the interval under consideration.

Lemma 15. *Let $\mathcal{A} > 0$ be a given constant and let $A(U_1)$ be the rational function of (8). Let g and G be the functions defined in (17) and (18) respectively. Suppose that*

- G and g are surjective functions,
- exists $d \in [0, 1)$ such that $(\ln(G_1^d(x)/g_2^2(x)))' \geq 0$ for all $x > 0$,
- $G''(x) \geq 0$ for all $x > 0$,

hold, where G_1 is the primitive function of g_1 such that $G_1(0) = 0$. Then there exists $M(\mathcal{A}) > 0$ that satisfies the following condition: for $n > 1 + M(\mathcal{A})/(2-2d)$ there exists $c := c(n, \mathcal{A}) > 0$ such that $A^{-1}((0, \mathcal{A}]) \cap \mathbb{R}_{>0} = (0, c]$.

Proof. By Theorem 14 we have that there exists $M(\mathcal{A}) > 0$ such that the condition $A'(u_1) > 0$ is satisfied for $n > 1 + M(\mathcal{A})/(2-2d)$ and $u_1 \in A^{-1}((0, \mathcal{A}]) \cap \mathbb{R}_{>0}$. Fix $n \geq 1 + M(\mathcal{A})/(2-2d)$. From Lemma 1 we deduce that $U_n(u_1)$ defines a bijective function in $\mathbb{R}_{>0}$ and that

$$A(u_1) = \left(\frac{h}{2} g_1(U_1(u_1)) + \sum_{k=2}^{n-1} h g_1(U_k(u_1)) + \frac{h}{2} g_1(U_n(u_1)) \right) / g_2(U_n(u_1)).$$

Since $0 < U_1(u_1) < \dots < U_n(u_1)$, we have the following inequalities:

$$\frac{h}{2} g(U_n(u_1)) \leq A(u_1) \leq g(U_n(u_1)).$$

Since $\lim_{u_1 \rightarrow 0^+} g(U_n(u_1)) = 0$, there exists $\epsilon > 0$ such that $(0, \epsilon] \subset A^{-1}((0, \mathcal{A}]) \cap \mathbb{R}_{>0}$. We claim that

$$(0, c_0] = A^{-1}((0, \mathcal{A}]) \cap \mathbb{R}_{>0}$$

with

$$c_0 := \sup\{\epsilon : (0, \epsilon] \subset A^{-1}((0, \mathcal{A}]) \cap \mathbb{R}_{>0}\}.$$

Indeed, from the definition of c_0 we obtain that $(0, c_0) \subset A^{-1}((0, \mathcal{A}]) \cap \mathbb{R}_{>0}$ and that $\lim_{u_1 \rightarrow c_0^-} A(u_1) = A(c_0) \leq \mathcal{A}$. Therefore, we deduce that

$$(0, c_0] \subset A^{-1}((0, \mathcal{A}]) \cap \mathbb{R}_{>0}.$$

We now show that the last set inclusion is an equality. Suppose that there exists $\delta > c_0$ such that $A(\delta) \leq \mathcal{A}$. Let $c_1 := \inf\{\delta : \delta > c_0, A(\delta) \leq \mathcal{A}\}$. From the definition of c_0 , the interval (c_0, c_1) is not empty. Since $A(x) > \mathcal{A}$ for all $x \in (c_0, c_1)$, we have that $A'(c_1) \leq 0$, which contradicts the fact that $A'(u_1) > 0$ for all $u_1 \in A^{-1}((0, \mathcal{A}]) \cap \mathbb{R}_{>0}$. \square

Now we state and prove the main result of this section:

Theorem 16. *Let $\alpha > 0$ be a given constant. Let g and G be the functions defined in (17) and (18) respectively. Suppose that*

- G and g are surjective functions,
- exists $d \in [0, 1)$ such that $(\ln(G_1^d(x)/g_2^2(x)))' \geq 0$ for all $x > 0$,
- $G''(x) \geq 0$ for all $x > 0$,

hold, where G_1 is the primitive function of g_1 such that $G_1(0) = 0$. Then there exists $M(\alpha) > 0$ such that (6) has a unique positive solution for $n > 1 + M(\alpha)/(2 - 2d)$.

Proof. Proposition 13 shows that (6) has solutions in $(\mathbb{R}_{>0})^n$ for any $\alpha > 0$ and any $n \in \mathbb{N}$. Therefore, there remains to show the uniqueness assertion.

By Theorem 14 we have that there exists $M(\mathcal{A}) > 0$ such that the condition $A'(u_1) > 0$ is satisfied for $n > 1 + M(\mathcal{A})/(2 - 2d)$ and $u_1 \in A^{-1}((0, \mathcal{A}]) \cap \mathbb{R}_{>0}$. From Lemma 15, there exists $c = c(n, \alpha)$ such that $A^{-1}((0, \alpha]) \cap \mathbb{R}_{>0} = (0, c]$. Arguing by contradiction, assume that there exist two distinct positive solutions (u_1, \dots, u_n) , $(\hat{u}_1, \dots, \hat{u}_n) \in (\mathbb{R}_{>0})^n$ of (6). This implies that $u_1 \neq \hat{u}_1$ and $A(u_1) = A(\hat{u}_1)$, where $A(U_1)$ is defined in (8). But this contradicts the fact that $A'(u_1) > 0$ holds in $(0, c]$, showing thus the theorem. \square

3. NUMERICAL CONDITIONING

Let be given $n \in \mathbb{N}$ and $\alpha^* > 0$. Let g and G be the functions defined in (17) and (18) respectively. Suppose that

- G and g are surjective functions,
- exists $d \in [0, 1)$ such that $(\ln(G_1^d(x)/g_2^2(x)))' \geq 0$ for all $x > 0$,
- $G''(x) \geq 0$ for all $x > 0$,

hold, where G_1 is the primitive function of g_1 such that $G_1(0) = 0$. In order to compute the positive solution of (7) for this value of n and $A = \alpha^*$, we shall consider (7) as a family of systems parametrized by the values α of A , following the positive real path determined by (7) when A runs through a suitable interval whose endpoints are α_* and α^* , where α_* be a positive constant independent of h to be fixed in Section 4.

A critical measure for the complexity of this procedure is the condition number of the path considered, which is essentially determined by the inverse of the Jacobian matrix of (7) with respect to the variables U_1, \dots, U_n , and the gradient vector of (7) with respect to the variable A on the path. In this section we prove the invertibility of such Jacobian matrix, and obtain an explicit form of its inverse. Then we obtain an upper bound on the condition number of the path under consideration.

Let $F := F(A, U) : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$ be the nonlinear map defined by the right-hand side of (7). In this section we analyze the invertibility of the Jacobian matrix of F with respect to the variables U , namely,

$$J(A, U) := \frac{\partial F}{\partial U}(A, U) := \begin{pmatrix} \Gamma_1 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & \Gamma_n \end{pmatrix},$$

with $\Gamma_1 := 1 + \frac{1}{2}h^2g_1'(U_1)$, $\Gamma_i := 2 + h^2g_1'(U_i)$ for $2 \leq i \leq n-1$ and $\Gamma_n := 1 + \frac{1}{2}h^2g_1'(U_n) - hAg_2'(U_n)$.

We start relating the nonsingularity of the Jacobian matrix $J(\alpha, u)$ with that of the corresponding point in the path determined by (7). Let $(\alpha, u) \in (\mathbb{R}_{>0})^{n+1}$ be a solution of (7) for $A = \alpha$. Taking derivatives with respect to U_1 in (8) and substituting u_1 for U_1 we obtain the following tridiagonal system:

$$\begin{pmatrix} \Gamma_1(u_1) & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & \Gamma_n(u_1) \end{pmatrix} \begin{pmatrix} 1 \\ U_2'(u_1) \\ \vdots \\ U_n'(u_1) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ hg_2(U_n(u_1))A'(u_1) \end{pmatrix}.$$

For $1 \leq k \leq n-1$, we denote by $\Delta_k := \Delta_k(A, U)$ the k th principal minor of the matrix $J(A, U)$, that is, the $(k \times k)$ -matrix formed by the first k rows and the first k columns of $J(A, U)$. By the Cramer rule we deduce the identities:

$$(31) \quad hg_2(U_n(u_1))A'(u_1) = \det(J(\alpha, u)),$$

$$(32) \quad \det(J(\alpha, u))U_k'(u_1) = hg_2(U_n(u_1))A'(u_1)\det(\Delta_{k-1}(\alpha, u)),$$

for $2 \leq k \leq n$.

Let $\alpha > 0$ be a given constant. Then Theorem 14 asserts that $A'(u_1) > 0$ holds. Combining this inequality with (31) we conclude that $\det(J(\alpha, u)) > 0$ holds. Furthermore, by (32), we have

$$(33) \quad U_k'(u_1) = \det(\Delta_{k-1}(\alpha, u)) \quad (2 \leq k \leq n).$$

Combining Remark 1(iv) and (33) it follows that $\det(\Delta_k(\alpha, u)) > 0$ holds for $1 \leq k \leq n-1$. As a consequence, we have that all the principal minors of the symmetric matrix $J(\alpha, u)$ are positive. Then the Sylvester criterion shows that $J(\alpha, u)$ is positive definite. These remarks allows us to prove the following result.

Theorem 17. *Let $(\alpha, v) \in (\mathbb{R}_{>0})^{n+1}$ be a solution of (7) for $A = \alpha$. Let g and G be the functions defined in (17) and (18) respectively. Suppose that*

- G and g are surjective functions,
- exists $d \in [0, 1)$ such that $(\ln(G_1^d(x)/g_2^2(x)))' \geq 0$ for all $x > 0$,
- $G''(x) \geq 0$ for all $x > 0$,

hold, where G_1 is the primitive function of g_1 such that $G_1(0) = 0$. Then there exists $M(\alpha) > 0$ such that the matrix $J(\alpha, u)$ is symmetric and positive definite for $n > 1 + M(\alpha)/(2 - 2d)$.

Having shown the invertibility of the matrix $J(\alpha, u)$ for every solution $(\alpha, u) \in (\mathbb{R}_{>0})^{n+1}$ of (7), the next step is to obtain explicitly the corresponding inverse matrices $J^{-1}(\alpha, u)$. For this purpose, we establish a result on the structure of the matrix $J^{-1}(\alpha, u)$.

Proposition 18. *Let $(\alpha, u) \in (\mathbb{R}_{>0})^{n+1}$ be a solution of (7). Let g and G be the functions defined in (17) and (18) respectively. Suppose that*

- G and g are surjective functions,
- exists $d \in [0, 1)$ such that $(\ln(G_1^d(x)/g_2^2(x)))' \geq 0$ for all $x > 0$,
- $G'''(x) \geq 0$ for all $x > 0$,

hold, where G_1 is the primitive function of g_1 such that $G_1(0) = 0$. Then there exists $M(\alpha) > 0$ such that the following matrix factorization holds:

$$J^{-1}(\alpha, u) = \begin{pmatrix} 1 & \frac{1}{u'_2} & \frac{1}{u'_3} & \cdots & \frac{1}{u'_n} \\ & 1 & \frac{u'_2}{u'_3} & \cdots & \frac{u'_2}{u'_n} \\ & & \ddots & \ddots & \vdots \\ & & & 1 & \frac{u'_{n-1}}{u'_n} \\ & & & & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{u'_2} & & & & \\ \frac{1}{u'_3} & \frac{u'_2}{u'_3} & & & \\ \vdots & \vdots & \ddots & & \\ \frac{1}{u'_n} & \frac{u'_2}{u'_n} & \cdots & \frac{u'_{n-1}}{u'_n} & \\ \frac{1}{d(J)} & \frac{u'_2}{d(J)} & \cdots & \frac{u'_{n-1}}{d(J)} & \frac{u'_n}{d(J)} \end{pmatrix},$$

for $n > 1 + M(\alpha)/(2 - 2d)$, where $d(J) := \det(J(\alpha, u))$ and $u'_k := U'_k(u_1)$ for $2 \leq k \leq n$.

Proof. Since $J(\alpha, u)$ is symmetric, invertible, tridiagonal and their $(n - 1)$ th principal minor is positive definite, the proof follows in the same way as that of [DM09, Proposition 25]. \square

From the explicitation of the inverse of the Jacobian matrix $J(A, U)$ on the points of the real path determined by (7), we can finally obtain estimates on the condition number of such a path.

Let $\alpha^* > 0$ and $\alpha_* > 0$ constants independents of h be given. Then Theorem 16 proves that (7) has a unique positive solution with $A = \alpha$ for every α in the real interval $\mathcal{I} := \mathcal{I}(\alpha_*, \alpha^*)$ whose endpoints are α_* and α^* , which we denote by $(u_1(\alpha), U_2(u_1(\alpha)), \dots, U_n(u_1(\alpha)))$. We bound the condition number

$$\kappa := \max\{\|\varphi'(\alpha)\|_\infty : \alpha \in \mathcal{I}\},$$

associated to the function $\varphi : \mathcal{I} \rightarrow \mathbb{R}^n$, $\varphi(\alpha) := (u_1(\alpha), U_2(u_1(\alpha)), \dots, U_n(u_1(\alpha)))$.

For this purpose, from the Implicit Function Theorem we have

$$\begin{aligned} \|\varphi'(\alpha)\|_\infty &= \left\| \left(\frac{\partial F}{\partial U}(\alpha, \varphi(\alpha)) \right)^{-1} \frac{\partial F}{\partial A}(\alpha, \varphi(\alpha)) \right\|_\infty \\ &= \left\| J^{-1}(\alpha, \varphi(\alpha)) \frac{\partial F}{\partial A}(\alpha, \varphi(\alpha)) \right\|_\infty. \end{aligned}$$

We observe that $(\partial F / \partial A)(\alpha, \varphi(\alpha)) = \left(0, \dots, 0, -hg_2(U_n(u_1(\alpha))) \right)^t$ holds. From Proposition 18 we obtain

$$\|\varphi'(\alpha)\|_\infty = \left\| \frac{hg_2(U_n(u_1(\alpha)))}{\det(J(\alpha, \varphi(\alpha)))} \left(1, U'_2(u_1(\alpha)), \dots, U'_n(u_1(\alpha)) \right)^t \right\|_\infty.$$

Combining this identity with (31), we conclude that

$$\|\varphi'(\alpha)\|_\infty = \left\| \frac{1}{A'(u_1(\alpha))} \left(1, U'_2(u_1(\alpha)), \dots, U'_n(u_1(\alpha)) \right)^t \right\|_\infty.$$

From Lemma 1, we deduce the following proposition.

Proposition 19. *Let $\alpha^* > 0$ and $\alpha_* > 0$ constants independents of h be given. Let g and G be the functions defined in (17) and (18) respectively. Suppose that*

- G and g are surjective functions,
- exists $d \in [0, 1)$ such that $(\ln(G_1^d(x)/g_2^2(x)))' \geq 0$ for all $x > 0$,
- $G'''(x) \geq 0$ for all $x > 0$,

hold, where G_1 is the primitive function of g_1 such that $G_1(0) = 0$. Then there exists $M(\mathcal{I}) > 0$ such that

$$\|\varphi'(\alpha)\|_\infty = \frac{U'_n(u_1(\alpha))}{A'(u_1(\alpha))}$$

holds for $\alpha \in \mathcal{I}$ and $n > 1 + M(\mathcal{I})/(2 - 2d)$.

Combining Proposition 19 and (28) we conclude that

$$\|\varphi'(\alpha)\|_\infty < \frac{4G_1(U_n(u_1(\alpha)))}{(1 - d)\alpha g_1(U_n(u_1(\alpha)))}.$$

Applying Lemma 9 and Proposition 11 we deduce the following result.

Theorem 20. *Let $\alpha^* > 0$ and $\alpha_* > 0$ constants independent of h be given. Let g and G be the functions defined in (17) and (18) respectively. Suppose that*

- G and g are surjective functions,
- exists $d \in [0, 1)$ such that $(\ln(G_1^d(x)/g_2^2(x)))' \geq 0$ for all $x > 0$,
- $G''(x) \geq 0$ for all $x > 0$,

hold, where G_1 is the primitive function of g_1 such that $G_1(0) = 0$. Then there exists a constant $\kappa_1(\alpha_*, \alpha^*) > 0$ independent of h such that

$$\kappa < \kappa_1(\alpha_*, \alpha^*).$$

4. AN EFFICIENT NUMERICAL ALGORITHM

As a consequence of the well conditioning of the positive solutions of (7), we shall exhibit an algorithm computing the positive solution of (7) for $A = \alpha^*$. This algorithm is a homotopy continuation method (see, e.g., [OR70, §10.4], [BCSS98, §14.3]) having a cost which is *linear* in n .

There are two different approaches to estimate the cost of our procedure: using Kantorovich-type estimates as in [OR70, §10.4], and using Smale-type estimates as in [BCSS98, §14.3]. We shall use the former, since we are able to control the condition number in suitable neighborhoods of the real paths determined by (7). Furthermore, the latter does not provide significantly better estimates.

Let $\alpha_* > 0$ be a constant independent of h . Let g and G be the functions defined in (17) and (18) respectively. Suppose that

- G and g are surjective functions,
- exists $d \in [0, 1)$ such that $(\ln(G_1^d(x)/g_2^2(x)))' \geq 0$ for all $x > 0$,
- $G''(x) \geq 0$ and $g''(x) \geq 0$ for all $x > 0$,

hold, where G_1 is the primitive function of g_1 such that $G_1(0) = 0$. Then the path defined by the positive solutions of (7) with $\alpha \in [\alpha_*, \alpha^*]$ is smooth, and the estimate of Theorem 20 hold. Assume that we are given a suitable approximation $u^{(0)}$ of the positive solution $\varphi(\alpha_*)$ of (7) for $A = \alpha_*$. In this section we exhibit an algorithm which, on input $u^{(0)}$, computes an approximation of $\varphi(\alpha^*)$. We recall that φ denotes the function which maps each $\alpha > 0$ to the positive solution of (7) for $A = \alpha$. More precisely $\varphi : [\alpha_*, \alpha^*] \rightarrow \mathbb{R}^n$ is the function which maps each $\alpha \in [\alpha_*, \alpha^*]$ to the positive solution of (7) for $A = \alpha$, namely

$$\varphi(\alpha) := (u_1(\alpha), \dots, u_n(\alpha)) := (u_1(\alpha), U_2(u_1(\alpha)), \dots, U_n(u_1(\alpha))).$$

From Lemma 12 and Lemma 9, we have that the coordinates of the positive solution of (7) tend to zero when α tends to zero. Therefore, for α small enough,

we obtain a suitable approximation of the positive solution (7) for $A = \alpha_*$, and we track the positive real path determined by (7) until $A = \alpha^*$.

Let $0 < \alpha_* < \alpha^*$ be a constant independent of h to be determined. Fix $\alpha \in [\alpha_*, \alpha^*]$. By Lemma 12 it follows that $\varphi(\alpha)$ is an interior point of the compact set

$$K_\alpha := \{u \in \mathbb{R}^n : \|u\|_\infty \leq 2C_2(\alpha)\},$$

where

$$C_2(\alpha) := G^{-1}\left(G\left(g^{-1}\left(\alpha\widehat{C}(\alpha)\right)\right) + \frac{\alpha^2}{2}\right),$$

with

$$\widehat{C}(\alpha) := 1 + \frac{g_2'(C_1(\alpha))\alpha^2}{2g_2(g^{-1}(\alpha)/e^M)G'(g^{-1}(\alpha)/e^M)},$$

$$C_1(\alpha) := G^{-1}\left(G\left(g^{-1}\left(\frac{\alpha}{\sqrt{1-d}}\right)\right) + \frac{\alpha^2}{2}\right),$$

and $M := g_1'(C_1(\alpha))$.

First we prove that the Jacobian matrix $J_\alpha(u) := (\partial F/\partial U)(\alpha, u)$ is invertible in a suitable subset of K_α . Let $u \in \mathbb{R}^n$ and $v \in \mathbb{R}^n$ be points with

$$\|u - \varphi(\alpha)\|_\infty < \delta_\alpha, \quad \|v - \varphi(\alpha)\|_\infty < \delta_\alpha,$$

where $\delta_\alpha > 0$ is a constant to be determined. Note that if $\delta_\beta \leq C_2(\alpha)$ then $u \in K_\alpha$ and $v \in K_\alpha$. By the Mean Value Theorem, we see that the entries of the diagonal matrix $J_\alpha(u) - J_\alpha(v)$ satisfy the estimates

$$\begin{aligned} \left| (J_\alpha(u) - J_\alpha(v))_{ii} \right| &\leq 2h^2 g_1''(2C_2(\alpha)) \delta_\alpha, \quad (1 \leq i \leq n-1) \\ \left| (J_\alpha(u) - J_\alpha(v))_{nn} \right| &\leq 2h \max\{\alpha g_2''(2C_2(\alpha)), g_1''(2C_2(\alpha))\} \delta_\alpha. \end{aligned}$$

By Theorem 17 and Proposition 18 we have that the matrix $J_{\varphi(\alpha)} := J_\alpha(\varphi(\alpha)) = (\partial F/\partial U)(\alpha, \varphi(\alpha))$ is invertible and

$$(J_{\varphi(\alpha)}^{-1})_{ij} = \sum_{k=\max\{i,j\}}^{n-1} \frac{U_i'(u_1(\alpha))U_j'(u_1(\alpha))}{U_k'(u_1(\alpha))U_{k+1}'(u_1(\alpha))} + \frac{U_i'(u_1(\alpha))U_j'(u_1(\alpha))}{U_n'(u_1(\alpha))\det(J_{\varphi(\alpha)})}$$

holds for $1 \leq i, j \leq n$. According to Lemma 1, we have $U_n'(u_1(\alpha)) \geq \dots \geq U_2'(u_1(\alpha)) \geq 1$. These remarks show that

$$\begin{aligned} &\left\| J_{\varphi(\alpha)}^{-1}(J_\alpha(u) - J_\alpha(v)) \right\|_\infty \leq \\ (34) \quad &\leq \eta_\alpha \delta_\alpha \left(2 + \frac{h^2 + \sum_{j=2}^{n-1} h^2 U_j'(u_1(\alpha)) + h U_n'(u_1(\alpha))}{|\det(J_{\varphi(\alpha)})|} \right) \\ &\leq 2\eta_\alpha \delta_\alpha \left(1 + \frac{h U_n'(u_1(\alpha))}{|\det(J_{\varphi(\alpha)})|} \right), \end{aligned}$$

where $\eta_\alpha := 2 \max\{g_1''(2C_2(\alpha)), \alpha g_2''(2C_2(\alpha))\}$. From (31), we obtain the following identity:

$$\frac{h U_n'(u_1(\alpha))}{|\det(J_{\varphi(\alpha)})|} = \frac{U_n'(u_1(\alpha))}{A'(u_1(\alpha))g_2(u_n(\alpha))}.$$

From (28), we have that

$$(35) \quad \frac{hU'_n(u_1(\alpha))}{|\det(J_{\varphi(\alpha)})|} = \frac{U'_n(u_1(\alpha))}{A'(u_1(\alpha))g_2(u_n(\alpha))} \leq \frac{4G_1(u_n(\alpha))}{(1-d)g_1(u_n(\alpha))g_2(u_n(\alpha))A(u_1(\alpha))}.$$

From Lemma 3, we have that $G'(x) > 0$ and $g'(x) > 0$ in $\mathbb{R}_{>0}$. Since G is an increasing function, we deduce that

$$\frac{G_1(u_n(\alpha))}{g_1(u_n(\alpha))g_2(u_n(\alpha))} < \frac{1}{2g'_2(u_n(\alpha))}.$$

Combining the last inequality with (34) and (35), we obtain

$$\left\| J_{\varphi(\alpha)}^{-1}(J_\alpha(u) - J_\alpha(v)) \right\|_\infty \leq 2\eta_\alpha \delta_\alpha \left(1 + \frac{2}{(1-d)g'_2(u_n(\alpha))A(u_1(\alpha))} \right).$$

From Corollary 7, we have that

$$(36) \quad \left\| J_{\varphi(\alpha)}^{-1}(J_\alpha(u) - J_\alpha(v)) \right\|_\infty \leq \left(\frac{4\eta_\alpha(\theta^* + 1)}{g'_2(g^{-1}(\alpha))(1-d)\alpha} \right) \delta_\alpha.$$

with $\theta^* := g'_2(g^{-1}(\alpha^*))(1-d)\alpha^*/2$. Hence, defining δ_β in the following way:

$$(37) \quad \delta_\alpha := \min \left\{ \frac{g'_2(g^{-1}(\alpha))(1-d)\alpha}{16\eta_\alpha(\theta^* + 1)}, C_2(\alpha) \right\},$$

we obtain

$$(38) \quad \left\| J_{\varphi(\alpha)}^{-1}(J_\alpha(u) - J_\alpha(v)) \right\|_\infty \leq \frac{1}{4}.$$

In particular, for $v = \varphi(\alpha)$, this bound allows us to consider $J_\alpha(u)$ as a perturbation of $J_{\varphi(\alpha)}$. More precisely, by a standard perturbation lemma (see, e.g., [OR70, Lemma 2.3.2]) we deduce that $J_\alpha(u)$ is invertible for every $u \in \mathcal{B}_{\delta_\alpha}(\varphi(\alpha))$ and we obtain the following upper bound:

$$(39) \quad \left\| J_\alpha(u)^{-1}J_{\varphi(\alpha)} \right\|_\infty \leq \frac{4}{3}.$$

In order to describe our method, we need a sufficient condition for the convergence of the standard Newton iteration associated to (7) for any $\alpha \in [\alpha_*, \alpha^*]$. Arguing as in [OR70, 10.4.2] we deduce the following remark, which in particular implies that the Newton iteration under consideration converges.

Remark 21. Set $\delta := \min\{\delta_\alpha : \alpha \in [\alpha_*, \alpha^*]\}$. Fix $\alpha \in [\alpha_*, \alpha^*]$ and consider the Newton iteration

$$u^{(k+1)} = u^{(k)} - J_\alpha(u^{(k)})^{-1}F(\alpha, u^{(k)}) \quad (k \geq 0),$$

starting at $u^{(0)} \in K_\alpha$. If $\|u^{(0)} - \varphi(\alpha)\|_\infty < \delta$, then

$$\|u^{(k)} - \varphi(\alpha)\|_\infty < \frac{\delta}{3^k}$$

holds for $k \geq 0$.

Now we can describe our homotopy continuation method. Let $\alpha_0 := \alpha_* < \alpha_1 < \dots < \alpha_N := \alpha^*$ be a uniform partition of the interval $[\alpha_*, \alpha^*]$, with N to be fixed. We define an iteration as follows:

$$(40) \quad u^{(k+1)} = u^{(k)} - J_{\alpha_k}(u^{(k)})^{-1}F(\alpha_k, u^{(k)}) \quad (0 \leq k \leq N-1),$$

$$(41) \quad u^{(N+k+1)} = u^{(N+k)} - J_{\alpha^*}(u^{(N+k)})^{-1}F(\alpha^*, u^{(N+k)}) \quad (k \geq 0).$$

In order to see that the iteration (40)–(41) yields an approximation of the positive solution $\varphi(\alpha^*)$ of (7) for $A = \alpha^*$, it is necessary to obtain a condition assuring that (40) yields an attraction point for the Newton iteration (41). This relies on a suitable choice for N , which we now discuss.

By Theorem 20, we have

$$\begin{aligned} \|\varphi(\alpha_{i+1}) - \varphi(\alpha_i)\|_\infty &\leq \max\{\|\varphi'(\alpha)\|_\infty : \alpha \in [\alpha_*, \alpha^*]\} |\alpha_{i+1} - \alpha_i| \\ &\leq \kappa_1 \frac{\alpha^*}{N}, \end{aligned}$$

for $0 \leq i \leq N-1$, where κ_1 is an upper bound of the condition number independent of h . Thus, for $N := \lceil 3\alpha^* \kappa_1 / \delta \rceil + 1 = O(1)$, by the previous estimate we obtain the following inequality:

$$(42) \quad \|\varphi(\alpha_{i+1}) - \varphi(\alpha_i)\|_\infty < \frac{\delta}{3}$$

for $0 \leq i \leq N-1$. Our next result shows that this implies the desired result.

Lemma 22. *Set $N := \lceil 3\alpha^* \kappa_1 / \delta \rceil + 1$. Then, for every $u^{(0)}$ with $\|u^{(0)} - \varphi(\alpha_*)\|_\infty < \delta$, the point $u^{(N)}$ defined in (40) is an attraction point for the Newton iteration (41).*

Proof. By hypothesis, we have $\|u^{(0)} - \varphi(\alpha_*)\|_\infty < \delta$. Arguing inductively, suppose that $\|u^{(k)} - \varphi(\alpha_k)\|_\infty < \delta$ holds for a given $0 \leq k < N$. By Remark 21 we have that $u^{(k)}$ is an attraction point for the Newton iteration associated to (7) for $A = \alpha_k$. Furthermore, Remark 21 also shows that $\|u^{(k+1)} - \varphi(\alpha_k)\|_\infty < \delta/3$ holds. Then

$$\begin{aligned} \|u^{(k+1)} - \varphi(\alpha_{k+1})\|_\infty &\leq \|u^{(k+1)} - \varphi(\alpha_k)\|_\infty + \|\varphi(\alpha_k) - \varphi(\alpha_{k+1})\|_\infty \\ &< \frac{1}{3}\delta + \frac{1}{3}\delta < \delta, \end{aligned}$$

where the inequality $\|\varphi(\alpha_{k+1}) - \varphi(\alpha_k)\|_\infty < \delta/3$ follows by (42). This completes the inductive argument and shows in particular that $u^{(N)}$ is an attraction point for the Newton iteration (41). \square

Next we consider the convergence of (41), starting with a point $u^{(N)}$ satisfying the condition $\|u^{(N)} - \varphi(\alpha^*)\|_\infty < \alpha \leq \delta_{\alpha^*}$. Combining this inequality with (37) we deduce that $u^{(N)} \in K_{\alpha^*}$. Furthermore, we see that

$$\begin{aligned} \|u^{(N+1)} - \varphi(\alpha^*)\|_\infty &= \|u^{(N)} - J_{\alpha^*}(u^{(N)})^{-1} F(\alpha^*, u^{(N)}) - \varphi(\alpha^*)\|_\infty \\ &= \left\| J_{\alpha^*}(u^{(N)})^{-1} (J_{\alpha^*}(u^{(N)})(u^{(N)} - \varphi(\alpha^*)) - F(\alpha^*, u^{(N)}) + F(\alpha^*, \varphi(\alpha^*))) \right\|_\infty \\ (43) \quad &\leq \|J_{\alpha^*}(u^{(N)})^{-1} J_{\varphi(\alpha^*)}\|_\infty \\ &\quad \left\| J_{\varphi(\alpha^*)}^{-1} (J_{\alpha^*}(u^{(N)})(u^{(N)} - \varphi(\alpha^*)) - F(\alpha^*, u^{(N)}) + F(\alpha^*, \varphi(\alpha^*))) \right\|_\infty \\ &\leq \|J_{\alpha^*}(u^{(N)})^{-1} J_{\varphi(\alpha^*)}\|_\infty \|J_{\varphi(\alpha^*)}^{-1} (J_{\alpha^*}(u^{(N)}) - J_{\alpha^*}(\xi))\|_\infty \|u^{(N)} - \varphi(\alpha^*)\|_\infty, \end{aligned}$$

where ξ is a point in the segment joining the points $u^{(N)}$ and $\varphi(\alpha^*)$. Combining (36) and (39) we deduce that

$$\begin{aligned} \|u^{(N+1)} - \varphi(\alpha^*)\|_\infty &< \frac{4}{3} \|J_{\varphi(\alpha^*)}^{-1} (J_{\alpha^*}(u^{(N)}) - J_{\alpha^*}(\xi))\|_\infty \delta_{\alpha^*} \\ &< \frac{4c}{3} \delta_{\alpha^*}^2 \leq \frac{1}{3} \delta_{\alpha^*}, \end{aligned}$$

holds, with $c := (4\eta_{\alpha^*}(\theta^* + 1)) / (g_2'(g^{-1}(\alpha^*))(1 - d)\alpha^*)$. By an inductive argument we conclude that the iteration (41) is well-defined and converges to the positive solution $\varphi(\alpha^*)$ of (7) for $A = \alpha^*$. Furthermore, we conclude that the point $u^{(N+k)}$,

obtained from the point $u^{(N)}$ above after k steps of the iteration (41), satisfies the estimate

$$\|u^{(N+k)} - \varphi(\alpha^*)\|_\infty \leq \hat{c} \left(\frac{4c}{3} \delta_{\alpha^*}\right)^{2^k} \leq \hat{c} \left(\frac{1}{3}\right)^{2^k},$$

with $\hat{c} := 3/4c$. Therefore, in order to obtain an ε -approximation of $\varphi(\alpha^*)$, we have to perform $\log_2 \log_3(3/4c\varepsilon)$ steps of the iteration (41). Summarizing, we have the following result.

Lemma 23. *Let $\varepsilon > 0$ be given. Then, for every $u^{(N)} \in (\mathbb{R}_{>0})^n$ satisfying the condition $\|u^{(N)} - \varphi(\alpha^*)\|_\infty < \delta$, the iteration (41) is well-defined and the estimate $\|u^{(N+k)} - \varphi(\alpha^*)\|_\infty < \varepsilon$ holds for $k \geq \log_2 \log_3(3/4c\varepsilon)$.*

Let $\varepsilon > 0$. Assume that we are given $u^{(0)} \in (\mathbb{R}_{>0})^n$ such that $\|u^{(0)} - \varphi(\alpha_*)\|_\infty < \delta$ holds. In order to compute an ε -approximation of the positive solution $\varphi(\alpha^*)$ of (7) for $A = \alpha^*$, we perform N iterations of (40) and $k_0 := \lceil \log_2 \log_3(3/4c\varepsilon) \rceil$ iterations of (41). From Lemmas 22 and 23 we conclude that the output $u^{(N+k_0)}$ of this procedure satisfies the condition $\|u^{(N+k_0)} - \varphi(\alpha^*)\|_\infty < \varepsilon$. Observe that the Jacobian matrix $J_\alpha(u)$ is tridiagonal for every $\alpha \in [\alpha_*, \alpha^*]$ and every $u \in K_\alpha$. Therefore, the solution of a linear system with matrix $J_\alpha(u)$ can be obtained with $O(n)$ flops. This implies that each iteration of both (40) and (41) requires $O(n)$ flops. In conclusion, we have the following result.

Proposition 24. *Let $\varepsilon > 0$ and $u^{(0)} \in (\mathbb{R}_{>0})^n$ with $\|u^{(0)} - \varphi(\alpha_*)\|_\infty < \delta$ be given, where δ is defined as in Remark 21. Then the output of the iteration (40)–(41) is an ε -approximation of the positive solution $\varphi(\alpha^*)$ of (7) for $A = \alpha^*$. This iteration can be computed with $O(Nn + k_0n) = O(n \log_2 \log_2(1/\varepsilon))$ flops.*

Finally, we exhibit a starting point $u^{(0)} \in (\mathbb{R}_{>0})^n$ satisfying the condition of Proposition 24. Let $\alpha_* > 0$ be a constant independent of h to be determined. We study the constant

$$\delta := \min\{\delta_\alpha : \alpha \in [\alpha_*, \alpha^*]\},$$

where

$$\delta_\alpha := \min\left\{\frac{g'_2(g^{-1}(\alpha))(1-d)\alpha}{16\eta_\alpha(\theta^*+1)}, C_2(\alpha)\right\},$$

with

$$\begin{aligned} C_2(\alpha) &:= G^{-1}\left(G\left(g^{-1}\left(\alpha\widehat{C}(\alpha)\right)\right) + \frac{\alpha^2}{2}\right), \\ \widehat{C}(\alpha) &:= 1 + \frac{g'_2(C_1(\alpha))\alpha^2}{2g_2(g^{-1}(\alpha)/e^M)G'(g^{-1}(\alpha)/e^M)}, \\ C_1(\alpha) &:= G^{-1}\left(G\left(g^{-1}\left(\frac{\alpha}{\sqrt{1-d}}\right)\right) + \frac{\alpha^2}{2}\right), \end{aligned}$$

and $M := g'_1(C_1(\alpha))$.

Since $\widehat{C}(\alpha) \geq 1$, we have that

$$\begin{aligned} \eta_\alpha &= 2 \max\{g''_1(2C_2(\alpha)), g''_2(2C_2(\alpha))\alpha\} \\ &\leq 2 \max\{g''_1(2C_2(\alpha)), g''_2(2C_2(\alpha))g(G^{-1}(G(g^{-1}(\alpha\widehat{C}(\alpha))))\}), \\ &\leq 2 \max\{g''_1(2C_2(\alpha)), g''_2(2C_2(\alpha))g(C_2(\alpha))\}, \end{aligned}$$

As g_1 and g_2 are analytic functions in $x = 0$, in a neighborhood of $0 \in \mathbb{R}^n$, we obtain the following estimate:

$$\eta_\alpha \leq 2 \max\{S_1(\alpha), S_2(\alpha)\} (C_2(\alpha))^{p-2},$$

where p is the multiplicity of 0 as a root of g_1 and S_i is an analytic function in $x = 0$ such that $\lim_{\alpha \rightarrow 0} S_i(\alpha) \neq 0$ for $i = 1, 2$. Taking into account that $\alpha \in (0, \alpha^*]$ holds, we conclude that there exists a constant $\eta^* > 0$, which depends only on α^* , with

$$\eta_\alpha \leq 2\eta^* (C_2(\alpha))^{p-2}$$

for all $\alpha \in (0, \alpha^*]$. Moreover, with a similar argument we deduce that there exists a constant $\vartheta^* > 0$, which depends only on α^* , such that

$$(44) \quad \delta_\alpha \geq \min \left\{ \frac{\vartheta^*(1-d)}{16\eta^*(\theta^*+1)} \left(\frac{g^{-1}(\alpha)}{C_2(\alpha)} \right)^{p-2}, \frac{C_2(\alpha)}{g^{-1}(\alpha)} \right\} g^{-1}(\alpha).$$

We claim that

$$(45) \quad \lim_{\alpha \rightarrow 0^+} \frac{C_2(\alpha)}{g^{-1}(\alpha)} = 1^+.$$

In fact, since we have $\widehat{C}(\alpha) \geq 1$ and g^{-1} is increasing, it follows that

$$(46) \quad \frac{C_2(\alpha)}{g^{-1}(\alpha)} \geq 1.$$

On the other hand, there exist $\xi_1 \in \left(G(g^{-1}(\alpha\widehat{C}(\alpha))), G(g^{-1}(\alpha\widehat{C}(\alpha))) + \frac{\alpha^2}{2} \right)$ and $\xi_2 \in (\alpha, \alpha\widehat{C}(\alpha))$ with

$$\begin{aligned} \frac{C_2(\alpha)}{g^{-1}(\alpha)} &= \frac{g^{-1}(\alpha\widehat{C}(\alpha)) + (G^{-1})'(\xi_1)\frac{\alpha^2}{2}}{g^{-1}(\alpha)} \\ &= \frac{g^{-1}(\alpha) + (g^{-1})'(\xi_2)\alpha(\widehat{C}(\alpha) - 1) + (G^{-1})'(\xi_1)\frac{\alpha^2}{2}}{g^{-1}(\alpha)} \\ &\leq 1 + \frac{\alpha(\widehat{C}(\alpha) - 1)}{g'(g^{-1}(\alpha))g^{-1}(\alpha)} + \frac{\alpha^2}{2G'(g^{-1}(\alpha\widehat{C}(\alpha)))g^{-1}(\alpha)} \\ (47) \quad &\leq 1 + \frac{g(g^{-1}(\alpha))(\widehat{C}(\alpha) - 1)}{g'(g^{-1}(\alpha))g^{-1}(\alpha)} + \frac{(g(g^{-1}(\alpha)))^2}{2G'(g^{-1}(\alpha))g^{-1}(\alpha)}. \end{aligned}$$

Since g_1 and g_2 are analytic functions in $x = 0$, we see that

$$\begin{aligned} \lim_{\alpha \rightarrow 0^+} \frac{g(g^{-1}(\alpha))(\widehat{C}(\alpha) - 1)}{g'(g^{-1}(\alpha))g^{-1}(\alpha)} &= 0, \\ \lim_{\alpha \rightarrow 0^+} \frac{(g(g^{-1}(\alpha)))^2}{2G'(g^{-1}(\alpha))g^{-1}(\alpha)} &= 0, \end{aligned}$$

Combining these remarks with (46) and (47) we immediately deduce (45).

Combining (44) with (45) we conclude that there exists a constant $C^* > 0$, which depends only on α^* , with

$$\delta_\alpha \geq C^* g^{-1}(\alpha).$$

Therefore,

$$(48) \quad \delta = \min\{\delta_\alpha : \alpha \in [\alpha_*, \alpha^*]\} \geq C^* g^{-1}(1/\alpha_*).$$

From Lemma 12 and Lemma 9, we have

$$\varphi(\alpha_*) \in [g^{-1}(\alpha_*)/e^M, C_2(\alpha_*)]^n.$$

Furthermore, by (45), we deduce that

$$(49) \quad \left(\frac{C_2(\alpha_*)e^M}{g^{-1}(\alpha_*)} - 1 \right) \frac{g^{-1}(\alpha_*)}{e^M} \leq \left(\frac{C_2(\alpha_*)e^M}{g^{-1}(\alpha_*)} - 1 \right) g^{-1}(\alpha_*) < C^* g^{-1}(\alpha_*).$$

holds for $\alpha_* > 0$ small enough. Combining this with (48), we conclude that

$$(50) \quad \|u - \varphi(\alpha_*)\|_\infty \leq C_2(\alpha_*) - g^{-1}(\alpha_*)/e^M < \delta$$

holds for all $u \in [g^{-1}(\alpha_*)/e^M, C_2(\alpha_*)]^n$. Thus, let $\alpha_* < \alpha^*$ satisfy (49). Then, for any $u^{(0)}$ in the hypercube $[g^{-1}(\alpha_*)/e^M, C_2(\alpha_*)]^n$, the inequality

$$\|u^{(0)} - \varphi(\alpha_*)\|_\infty < \delta$$

holds. Therefore, applying Proposition 24, we obtain our main result.

Theorem 25. *Let $\varepsilon > 0$ be given. Then we can compute an ε -approximation of the positive solution of (7) for $A = \alpha^*$ with $O(n \log_2 \log_2(1/\varepsilon))$ flops.*

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